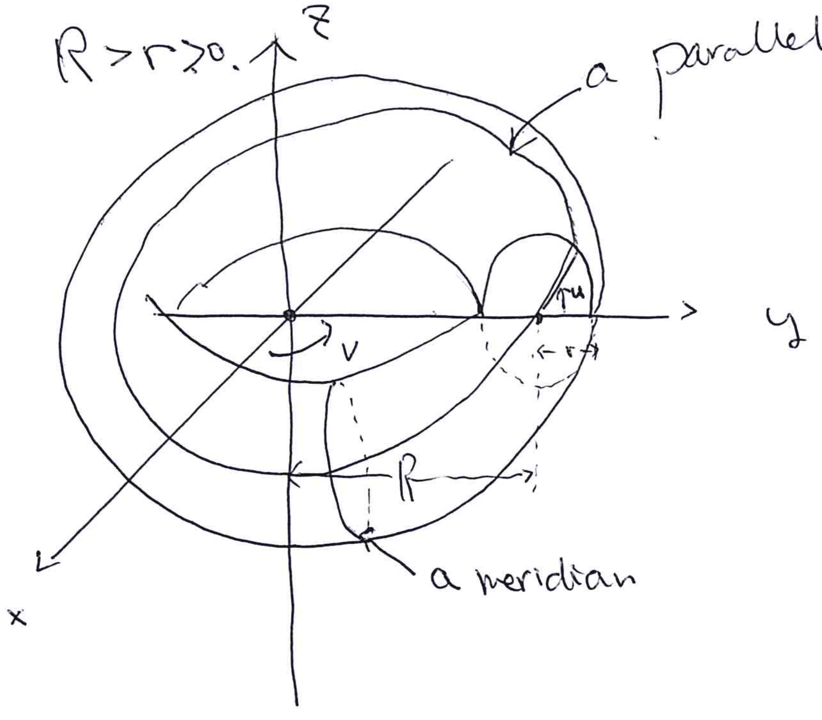
Tutorial 5

A line of curvature

Def: A curve $\alpha : I \to M$ ($\leftarrow$ a regular surface) is a line of curvature if $\alpha'(t)$ is a principal direction i.e. an eigenvector of the shape operator $S_{\alpha(t)}$ (at $\alpha(t)$, $\forall t \in I$).

e.g. Torus $X(u,v) = \big((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u$

$R > r > 0.$



meridian $= X(u, v_0)$, $v_0 =$ constant

parallel $= X(u_0, v)$, $u_0 =$ const

All meridians and parallels in the torus are lines of curvature.

To see it, we need to compute the matrix representation of the shape operator S under the basis $\{X_u, X_v\}$
$\{\overset{''}{X_1}, \overset{''}{X_2}\}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} = [h_{ij}]\cdot[g_{ij}]^{-1}$$

$$\boxed{\begin{cases} S(X_u) = a_{11} X_u + a_{12} X_v \\ S(X_v) = a_{21} X_u + a_{22} X_v \end{cases}}$$

$$X_u = (-r\sin u\cos v, -r\sin u\sin v, r\cos u)$$

$$X_v = (-(R+r\cos u)\sin v, (R+r\cos u)\cos v, 0)$$

$$X_{uu} = (-r\cos u\cos v, -r\cos u\sin v, -r\sin u)$$

$$X_{uv} = (r\sin u\sin v, -r\sin u\cos v, 0)$$

$$X_{vv} = (-(R+r\cos u)\cos v, -(R+r\cos u)\sin v, 0)$$

$$g_{11} = r^2\sin^2 u\cos^2 v + r^2\sin^2 u\sin^2 v + r^2\cos^2 u = r^2$$

$$g_{12} = 0$$

$$g_{22} = (R+r\cos u)^2$$

$$U = (-r(R+r\cos u)\cos v\cos u, -r(R+r\cos u)\sin v\cos u,$$
$$-r(R+r\cos u)\sin u) \cdot X \frac{1}{r(R+r\cos u)}$$
$$= (-\cos v\cos u, -\sin v\cos u, -\sin u)$$

$$h_{11} = \langle X_{uu}, N \rangle = r\cos^2 u \cos^2 v + r\sin^2 v \cos^2 u + r\sin^2 u$$

$$= r$$

$$h_{12} = \langle X_{uv}, N \rangle = 0$$

$$h_{22} = \langle X_{vv}, N \rangle = (R + r\cos u)\cos u$$

$$A = \begin{pmatrix} r & 0 \\ 0 & (R+r\cos u)\cos u \end{pmatrix} \begin{pmatrix} \frac{1}{r^2} & 0 \\ 0 & \frac{1}{(R+r\cos u)^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{\cos u}{R + r\cos u} \end{pmatrix}$$

$$\Rightarrow \begin{cases} S(X_u) = \frac{1}{r} X_u \\ S(X_v) = \frac{\cos u}{R + r\cos u} X_v \end{cases}$$

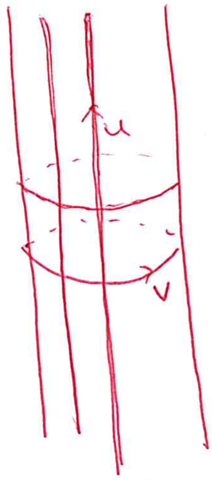
Therefore, all meridians and parallels of the torus are lines of curvature.

Thm: A regular parametrized surface without umbilic points can be reparametrized s.t the coordinate curves are lines of curvature. (See do Carmo's text-book P185 Corol. 4)

umbilic pt means the point s.t $k_1 = k_2 = \sqrt{K}$ i.e

all vectors are eigenvectors with eigenvalue $\sqrt{K}$.

e.g.



Cylinder

every pt $\begin{cases} k_1 = 0 \\ k_2 = -1 \end{cases}$

In such coordinate, $g_{12} = 0$ sine

$$k_1 \langle X_1, X_2 \rangle = \langle S(X_1), X_2 \rangle = \langle X_1, S(X_2) \rangle = k_2 \langle X_1, X_2 \rangle,$$

$$k_1 > k_2 \Rightarrow \langle X_1, X_2 \rangle = 0.$$

$$h_{11} = \langle S(X_1), X_1 \rangle = k_1 g_{11}$$

$$h_{12} = \langle S(X_1), X_2 \rangle = 0$$

$$h_{22} = \cdots = k_2 g_{22}$$

$$\boxed{\begin{aligned} (g_{ij}) &= \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix} \\ (h_{ij}) &= \begin{pmatrix} k_1 g_{11} & 0 \\ 0 & k_2 g_{22} \end{pmatrix} \end{aligned}}$$

Exercise: Surface of revolution parametrized by

$$X(u,v) = (g(u),\ h(u)\cos v,\ h(u)\sin v)$$

Then the meridian $X(u, \overset{\text{fix}}{v_0})$ and parallel $X(\overset{\text{fix}}{u_0}, v)$ are lines of curvature.

$$(g_{ij}) = \begin{pmatrix} g'^2 + h'^2 & 0 \\ 0 & h^2 \end{pmatrix}, \quad (h_{ij}) = \begin{pmatrix} \dfrac{g''h' - h''g'}{\sqrt{h'^2 + g'^2}} & 0 \\ 0 & \dfrac{hg'}{\sqrt{g'^2 + h'^2}} \end{pmatrix}$$

$$(a_{ij}) = \begin{pmatrix} \dfrac{g''h' - h''g'}{(g'^2 + h'^2)^{\frac{3}{2}}} & 0 \\ 0 & \dfrac{g'}{h\sqrt{g'^2 + h'^2}} \end{pmatrix}$$

A rmk for $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$

$$S(X_i) = \sum_{j=1}^{2} a_{ij} X_j, \quad i = 1, 2.$$

$$h_{ij} \triangleq \langle S(X_i), X_j \rangle = a_{ik} \langle X_k, X_j \rangle$$

$$= a_{ik}\, g_{kj}$$

$$\Rightarrow \quad g^{jl} h_{ij} = a_{ik} g_{kj} g^{jl}$$

Here $[g^{-1}]_{ij} = g^{ij}$ i.e. $g_{ij} g^{jk} = \delta_{ik}$

$$\Rightarrow \quad a_{il} = g^{lj} h_{ji} = h_{ij} g^{jl}.$$

$$\Rightarrow \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \cdot \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1}.$$